Solution to Assignment 6

Supplementary Problems

1. In the proof of Theorem 1 in Lecture 11, we only consider the case $R_j \in \mathcal{A}$, that is, $\partial f_1/\partial u \neq 0$ in R_j and leave out the case $\partial f_1/\partial v \neq 0$ on R_j. Provide a proof in this case. Suggestion: Switch the variables u and v.

   Solution. When $\partial f_1/\partial v \neq 0$ in R_j, define $\Psi = \Phi \circ T$ where $T(u,v) = (v,u)$. Then

$$(g_1(u,v), g_2(u,v)) = \Psi(u,v) = \Phi(v,u) = (f_1(v,u), f_2(v,u)) \ ,$$

   so $\partial g_1/\partial u = \partial f_1/\partial v \neq 0$ in $S_j = T^{-1}(R_j)$. Ψ is in Case (i) and the change of variables formula holds for Ψ. In other words,

$$\iint_{S_j} F(\Psi(u,v)) \left| \frac{\partial(g_1,g_2)}{\partial(u,v)} \right| dA(u,v) = \iint_{D_j} F(x,y)\, dA(x,y) \ .$$

   Now, using

$$\frac{\partial(g_1,g_2)}{\partial(u,v)} = -\frac{\partial(f_1,f_2)}{\partial(u,v)}$$

   we conclude the formula also holds for Φ.

2. Find the volume of the ball in $\mathbb{R}^4$, that is, $\{(x,y,z,w) : x^2+y^2+z^2+w^2 \leq R^2\}$. Suggestion: Apply the change of variables formula after introducing generalized polar coordinates $w = \rho\cos\psi, z = \rho\sin\psi\cos\varphi, x = \rho\sin\psi\sin\varphi\cos\theta, y = \rho\sin\psi\sin\varphi\sin\theta$ or use cross section method.

   Solution. A direct computation shows that

$$\frac{\partial(x,y,z,w)}{\partial(\rho,\psi,\varphi,\theta)} = \rho^3 \sin\psi \sin^2\varphi,$$

   so the volume of the 4-dim ball of radius R is given by

$$\iint_{B_R} 1\, dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{\pi} \frac{\partial(x,y,z,w)}{\partial(\rho,\psi,\varphi,\theta)}\, d\rho d\psi d\varphi d\theta = \frac{\pi^2}{2}R^4 \ .$$

   Alternatively, we use cross section method (Theorem 3 in Lecture 7). For $w \in [-R,R]$, the w-cross section is a 3-ball of radius $\sqrt{R^2-w^2}$, so its 3-dimensional area is given by $\frac{4}{3}\pi(R^2-w^2)^{3/2}$. The volume of B_R is

$$\int_{-R}^{R} \frac{4\pi}{3}(R^2-w^2)^{3/2} dw = \frac{\pi^2}{2}R^4 \ .$$